Notes on Thermal Equilibrium Fluctuations and Relaxation

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These notes are inspired by the lectures on Nonlinear Plasma Physics taught by Professor P. H. Diamond at UCSD. Throughout these notes I have used ideas gathered at the lectures and from various books and articles. The list of these books and articles can be found at the bibliography section.

1 Introduction

The Vlasov equation is obtained from the Liouville's equation using the BBGKY hierarchy and ignoring coupling of individual particles:

$$f(1,2) = f(1) f(2) + g(1,2) \approx f(1) f(2).$$

Thus the Vlasov theory is in a sense also fluid description, but fluid description in phase space. It neglects effects of the order $O\left(\frac{1}{n\lambda^3{}_D}\right)$ and corresponds to the system with

$$\begin{aligned} q &\to 0, \\ m &\to \infty, \\ n &\to \infty, \\ \frac{q}{m} &= const, \\ nq &= const. \end{aligned}$$

Thus the Vlasov theory lacks effect of discreetness of individual particles and coupling of individual particles to the radiation field. It means, in particular, that the Vlasov theory fails to answer the following questions:

1) What is the fluctuation spectrum of stable plasma?
2) What is radiation from plasma (rate and spectrum)?
3) What is cross section for scattering of radiation by plasma?

These questions can be answered by the test particle model, which is the main instrument of investigation of the questions of thermal equilibrium fluctuations and relaxation in these notes.

2 Fluctuation spectrum

Electric field fluctuation spectrum

Consider a test particle of charge q_T moving through uniform plasma. One can ask a question: What is the electric field (or potential) created by such particle? We already know the answer to this question for the case of motionless particle in a Maxwellian plasma – it is Debye potential:

$$\varphi(r) = \frac{q_T}{r} e^{-\frac{r}{\lambda_D}}. \tag{1}$$

Now let us turn to the derivation of the expression for electric field of moving charged particle in uniform plasma close to equilibrium. We consider only one discreet test particle in the bath of all plasma particles constituting the Vlasov fluid.
To the lowest order test particle moves along straight line:

$$\mathbf{r}_T = \mathbf{r}_{T0} + \mathbf{v}_T t. \tag{2}$$

Poisson's equation on its right hand side contains density of plasma given by (α is a type of plasma species)

$$\rho_{plasma} = \sum_\alpha n_\alpha q_\alpha \int f(\mathbf{v})\, d\mathbf{v}, \tag{3}$$

as well as density due to discreet test particle

$$\rho = q_T \delta(\mathbf{r} - \mathbf{r}_T) = q_T \delta(\mathbf{r} - \mathbf{r}_{T0} - \mathbf{v}_T t). \tag{4}$$

Adding to Poisson's equation the Vlasov equation for distribution function of plasma species we obtain the system

$$\begin{aligned} &\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\mathbf{E}\nabla_{\mathbf{v}} f_\alpha = 0, \\ &\nabla\mathbf{E} = 4\pi q_T \delta(\mathbf{r} - \mathbf{r}_{T_0} - \mathbf{v}_T t) + 4\pi\rho_{plasma}. \end{aligned} \tag{5}$$

We assume that plasma without test particle is uniform and field free, then linearize the above equations $f_\alpha(\mathbf{r},\mathbf{v},t) = f_{\alpha 0}(\mathbf{v}) + f_{\alpha 1}(\mathbf{r},\mathbf{v},t)$, where $f_{\alpha 1}(\mathbf{r},\mathbf{v},t)$ is perturbation in plasma due to the test charge. Test particle will attract particles of opposite sign and repel plasma particles of the same sign so that cloud of charged particles will surround our test particle shielding its electric potential. This cloud or "dress" of plasma around test particle will move with the particle, but it will not have enough time to symmetrically shield the potential.

The proper way to solve this problem is to use the Fourier transform in space and the Laplace transform in time. When we use the Laplace transform in time we essentially deal with initial value problem: test particle initially is at point $\mathbf{r}_{T_0}$ with velocity $\mathbf{v}_T$ and is not dressed: $f_{\alpha 1}(\mathbf{r},\mathbf{v},t=0) = 0$. However, for convenience, we will write all our results in the form of Fourier transform in time, all integrals that are ambiguous should be understood as taken on the Landau contour. So defining the direct and inverse Fourier transforms as

$$f_{\mathbf{k},\omega} = \iint f e^{-i\mathbf{kr}+i\omega t} d\mathbf{r} dt,$$
$$f = \iint f_{\mathbf{k},\omega} e^{i\mathbf{kr}-i\omega t} \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\omega}{2\pi}, \tag{6}$$

after simple algebra we obtain the electric field in the Fourier space:

$$\mathbf{E}_{\mathbf{k},\omega} = \frac{-\mathbf{k} 4\pi q_T e^{-i\mathbf{kr}_{T0}}}{(2\pi)^3 k^2 (\omega - \mathbf{kv}_T) \varepsilon(\mathbf{k},\omega)}, \tag{7}$$

where

$$\varepsilon(\mathbf{k},\omega) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \int \frac{\mathbf{k}\nabla_{\mathbf{v}} f_{\alpha 0}}{\omega - \mathbf{kv}} d\mathbf{v} \tag{8}$$

is the dielectric function of plasma (the integral should be evaluated using the Landau rule).

Going back to time space we get

$$\mathbf{E}_{\mathbf{k}}(t) = \int \mathbf{E}_{\mathbf{k},\omega} e^{-i\omega t} \frac{d\omega}{2\pi} =$$
$$= \frac{-i\mathbf{k} 4\pi q_T e^{-i\mathbf{kr}_{T0}} e^{-i\mathbf{kv}_T t}}{(2\pi)^3 k^2 \varepsilon(\mathbf{k},\mathbf{kv}_T)} + \sum_j \frac{-i\mathbf{k} 4\pi q_T e^{-i\mathbf{kr}_T} e^{-i\omega_j t}}{(2\pi)^3 k^2 (\omega_j - \mathbf{kv}_T) \left.\frac{d\varepsilon(\mathbf{k},\omega)}{d\omega}\right|_{\omega=\omega_j}}. \tag{9}$$

Here ω_j are the roots of the equation $\varepsilon(\mathbf{k},\omega) = 0$, but since we deal with plasma near equilibrium all ω_j must be Landau damped ($\operatorname{Im}\omega_{\mathrm{j}} < 0$). Thus we can ignore the sum over j, then

$$\mathbf{E}_{\mathbf{k}}(t) = \frac{-i\mathbf{k} 4\pi q_T e^{-i\mathbf{kr}_T}}{(2\pi)^3 k^2 \varepsilon(\mathbf{k},\mathbf{kv}_T)}. \tag{10}$$

From this we finally get the expression for the electric field of moving test particle in plasma

$$\mathbf{E}(\mathbf{r},t) = \int \frac{-i\mathbf{k} 4\pi q_T e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_T)}}{k^2 \varepsilon(\mathbf{k},\mathbf{kv}_T)} \frac{d\mathbf{k}}{(2\pi)^3}. \tag{11}$$

We see that electric field has the form of the Coulomb field of test particle divided by the velocity dependent dielectric function, so we say that electric field of test particle is dynamically screened by plasma.

Without proving let us just mention that for limiting cases we get expected results. For very slowly moving particle $\left(v_T \ll \sqrt{\frac{T_i}{m_i}}\right)$ electric potential $\varphi = -\nabla\mathbf{E}$ is the Debye potential

$$\varphi(\mathbf{r},t) \approx \frac{q_T}{|\mathbf{r}-\mathbf{r}_{T0}-\mathbf{v}_T t|} e^{-\frac{|\mathbf{r}-\mathbf{r}_{T0}-\mathbf{v}_T t|}{r_D}}, \tag{12}$$

while for particle moving very fast $\left(v_T \gg \sqrt{\frac{T_e}{m_e}}\right)$ there is no shielding at all:

$$\varphi \approx \frac{q_T}{|\mathbf{r}-\mathbf{r}_{T0}-\mathbf{v}_T t|}. \tag{13}$$

Before we have considered some "alien" test particle in plasma bath, but we can choose as test particle any of the plasma particles. There emerges double nature of a plasma particle in the test particle model: on the one hand it is discreet particle that excites oscillations through Cherenkov mechanics, on the other hand it is part of the Vlasov fluid that shields potential of the other individual particles and damps waves through Landau mechanism. In equilibrium the balance between emission of the wave and absorption is reached and some fluctuation spectrum is established. This feature of test particle model distinguishes it from Brownian motion where the role of test particle and external bath particles are clearly separated.

So we can consider plasma as an ensemble of test particles. It is reasonable to assume that these particles are uncorrelated in space and time, and distributed according to $f_{\alpha 0}(\mathbf{v})$ in velocity space. Then we can define the ensemble average value of some random variable A as

$$\langle A(\mathbf{r},\mathbf{v},t)\rangle = \sum_\alpha \int n_\alpha d\mathbf{r}' \int A(\mathbf{r},\mathbf{v},t;\mathbf{r}',\mathbf{v}') f_{\alpha 0}(\mathbf{v}') d\mathbf{v}'. \tag{14}$$

Electric field is a random in space and time variable. It is no surprise that average electric field in a uniform plasma is just zero $\langle \mathbf{E}(\mathbf{r},\mathbf{v},t)\rangle = 0$. However, $\langle \mathbf{E}^2(\mathbf{r},\mathbf{v},t)\rangle$ is not zero. Using equation (11) we can write

$$\begin{aligned}\langle \mathbf{E}^2(\mathbf{r},t)\rangle = \sum_\alpha \int n_\alpha d\mathbf{r}' \left(\int \frac{-i\mathbf{k}_1 4\pi q_\alpha e^{i\mathbf{k}_1(\mathbf{r}-\mathbf{r}')}}{(2\pi)^3 k_1^2 \varepsilon(\mathbf{k}_1,\mathbf{k}_1\mathbf{v}')} d\mathbf{k}_1 \right) \times \\ \left(\int \frac{i\mathbf{k}_2 4\pi q_\alpha e^{-i\mathbf{k}_2(\mathbf{r}-\mathbf{r}')}}{(2\pi)^3 k_2^2 \varepsilon^*(\mathbf{k}_2,\mathbf{k}_2\mathbf{v}')} d\mathbf{k}_2 \right) \int f_{\alpha 0}(\mathbf{v}') d\mathbf{v}'. \end{aligned} \tag{15}$$

Integrating over space we can get rid of one $\mathbf{k}$:

$$\int e^{i\mathbf{r}'(\mathbf{k}_2-\mathbf{k}_1)} \frac{d\mathbf{r}'}{(2\pi)^3} = \delta(\mathbf{k}_2-\mathbf{k}_1). \tag{16}$$

$$\langle \mathbf{E}^2(\mathbf{r},t)\rangle = \sum_\alpha n_\alpha (4\pi q_\alpha)^2 \iint \frac{\mathbf{k}\mathbf{k}}{k^4} \frac{f_{\alpha 0}(\mathbf{v}')}{|\varepsilon(\mathbf{k},\mathbf{k}\mathbf{v}')|^2} \frac{d\mathbf{k}}{(2\pi)^3} d\mathbf{v}'. \tag{17}$$

Because the dielectric function depends only on component of velocity along $\mathbf{k}$ we can integrate over other components of velocity:

$$F_\alpha(u) = \int f_{\alpha 0}(\mathbf{v})\, \delta\left(u - \frac{\mathbf{kv}}{k}\right) d\mathbf{v}. \tag{18}$$

Then defining $\omega = uk$ write

$$\left\langle \mathbf{E}^2(\mathbf{r},t)\right\rangle = \iint \sum_\alpha \frac{\mathbf{kk}\,(4\pi n_\alpha q_\alpha)^2\, \frac{2\pi}{n_\alpha} F_\alpha\left(\frac{\omega}{k}\right)}{k^5\, |\varepsilon(\mathbf{k},\omega)|^2} \frac{d\mathbf{k}}{(2\pi)^3}\frac{d\omega}{2\pi}, \tag{19}$$

$$\left\langle \mathbf{E}^2(\mathbf{r},t)\right\rangle = \iint \sum_\alpha \frac{(4\pi n_\alpha q_\alpha)^2\, \mathbf{kk} \frac{2\pi}{n_\alpha} \int f_{\alpha 0}(\mathbf{v})\, \delta(\omega - \mathbf{kv})\, d\mathbf{v}}{k^4\, |\varepsilon(\mathbf{k},\omega)|^2} \frac{d\mathbf{k}}{(2\pi)^3}\frac{d\omega}{2\pi}. \tag{20}$$

The fluctuation spectrum, by definition, is determined by

$$\left\langle \mathbf{E}^2(\mathbf{r},t)\right\rangle = \iint \left\langle \mathbf{E}^2(\mathbf{r},t)\right\rangle_{\mathbf{k},\omega} \frac{d\mathbf{k}}{(2\pi)^3}\frac{d\omega}{2\pi}. \tag{21}$$

Therefore fluctuation spectrum of the electric field in plasma is

$$\left\langle \mathbf{E}^2\right\rangle_{\mathbf{k},\omega} = \sum_\alpha \frac{\mathbf{kk}\,(4\pi n_\alpha q_\alpha)^2\, \frac{2\pi}{n_\alpha} F\left(\frac{\omega}{k}\right)}{k^5\, |\varepsilon(\mathbf{k},\omega)|^2} =$$
$$= \sum_\alpha (4\pi n_\alpha q_\alpha)^2 \frac{\mathbf{kk}}{k^4\, |\varepsilon(\mathbf{k},\omega)|^2} \int \frac{2\pi}{n_\alpha} f_{\alpha 0}(\mathbf{v})\, \delta(\omega - \mathbf{kv})\, d\mathbf{v}. \tag{22}$$

This fluctuation spectrum is determined by balance between excitation of electrostatic plasma waves through Cherenkov mechanism and damping of these waves due to Landau damping. So far we have considered plasma without magnetic field, for which phase velocities are higher than speed of light. Since test particle cannot move with a superluminal speed, it cannot excite Cherenkov electromagnetic radiation in such plasma. However, there will be electromagnetic radiation even in a plasma without magnetic field. We can show it by noticing that motion of test particle is equivalent to current $\mathbf{j} = q_T \mathbf{v} \delta(\mathbf{r} - \mathbf{r}_{T0} - \mathbf{v}_T t)$. This radiation is not Cherenkov radiation but so-called bremsstrahlung radiation, it appears in higher order then electrostatic and hence can be ignored. In plasma with magnetic field the situation is different and in such plasma test particles excite the electromagnetic Cherenkov radiation, which changes the expression for fluctuation spectrum.

Energy density

Having found the electric field fluctuation spectrum we can also write down the expression for energy density:

$$W = \frac{\left\langle \mathbf{E}^2\right\rangle}{8\pi} = \iint W_{\mathbf{k},\omega} \frac{d\mathbf{k}}{(2\pi)^3}\frac{d\omega}{2\pi}, \tag{23}$$

$$W_{\mathbf{k},\omega} = \sum_{\alpha} \frac{4\pi^2 n_\alpha q_\alpha^2 F_\alpha\left(\frac{\omega}{k}\right)}{k^3 \left|\varepsilon\left(\mathbf{k},\omega\right)\right|^2} = \sum_{\alpha} \frac{\pi m_\alpha \omega_{p\alpha}^2 F_\alpha\left(\frac{\omega}{k}\right)}{k^3 \left|\varepsilon\left(\mathbf{k},\omega\right)\right|^2}. \tag{24}$$

It is interesting to look at the energy density for different ranges of wavelengths. Defining

$$W = \frac{\left\langle \mathbf{E}^2 \right\rangle}{8\pi} = \iint W_{\mathbf{k}} \frac{d\mathbf{k}}{\left(2\pi\right)^3}, \tag{25}$$

and writing the dielectric function in the form

$$\varepsilon\left(\mathbf{k},\omega\right) = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \int \frac{\frac{\partial F_\alpha(u)}{\partial u}}{\frac{\omega}{k} - u} du = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} ⨍ \frac{\frac{\partial F_\alpha(u)}{\partial u}}{\frac{\omega}{k} - u} du - i\pi \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \left.\frac{\partial F_\alpha\left(u\right)}{\partial u}\right|_{u=\frac{\omega}{k}}, \tag{26}$$

$$\left|\varepsilon\right|^2 = \varepsilon_r^2 + \varepsilon_i^2 = \left(1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} ⨍ \frac{\frac{\partial F_\alpha(u)}{\partial u}}{\frac{\omega}{k} - u} du\right)^2 + \left(\pi \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \left.\frac{\partial F_\alpha\left(u\right)}{\partial u}\right|_{u=\frac{\omega}{k}}\right)^2 \tag{27}$$

we can perform integration over ω in some limiting cases.

In the limit of a Maxwellian plasma with $T_i = T_e$ electric energy per wavenumber is

$$W_k = \frac{1}{2} \frac{T_e}{1 + k^2 \lambda_D^2}. \tag{28}$$

For large wavelengths ($k\lambda_D \ll 1$) it yields

$$W_k = \frac{T_e}{2}, \tag{29}$$

i.e. electric field energy is equipartitioned and distributed according to the conventional rule of $\frac{T}{2}$ per degree of freedom. While for short wavelengths ($k\lambda_D \gg 1$) it yields

$$W_k = \frac{T_e}{2k^2 \lambda_D^2}. \tag{30}$$

We see that large-wavelength waves have much higher level of excitation than short-wavelength waves. It is expected because large-wavelength waves are weakly damped normal modes of plasma.

The fluctuation-dissipation theorem

It is worth nothing that the fluctuation spectrum obtained through the test particle model is in agreement with the fluctuation-dissipation theorem. It can be easily demonstrated for a Maxwellian plasma.

Indeed, from (26) imaginary part of the dielectric function is

$$\operatorname{Im}\varepsilon(\mathbf{k},\omega) = -\pi\sum_{\alpha}\frac{\omega_{p\alpha}^{2}}{k^{2}}\left.\frac{\partial F_{\alpha}(u)}{\partial u}\right|_{u=\frac{\omega}{k}}. \tag{31}$$

For a Maxwellian plasma

$$F'(u) = -\frac{mu}{T}F(u), \tag{32}$$

and the dielectric function becomes

$$\operatorname{Im}\varepsilon(\mathbf{k},\omega) = \pi\sum_{\alpha}\frac{m_{\alpha}\omega\omega_{p\alpha}^{2}}{Tk^{3}}F_{\alpha}\left(\frac{\omega}{k}\right). \tag{33}$$

From this we can write the fluctuation spectrum as

$$W_{\mathbf{k},\omega} = \frac{\left\langle \mathbf{E}^{2}(\mathbf{r},t)\right\rangle_{\mathbf{k},\omega}}{8\pi} = \frac{T}{\omega}\frac{\operatorname{Im}\varepsilon(\mathbf{k},\omega)}{\left|\varepsilon(\mathbf{k},\omega)\right|^{2}}, \tag{34}$$

which is exactly as required by the fluctuation-dissipation theorem.

3 Relaxation

Relaxation of distribution function

Relaxation equation describes the dynamics of the distribution function. It allows us to determine the characteristic time of relaxation of the distribution function as well as find macroscopic transport coefficients such as electric resistivity, for example.

We can obtain relaxation equation using the test particle model by finding the coefficients of the Fokker-Planck equation:

$$\frac{\partial f_{\alpha}}{\partial t} = -\frac{\partial}{\partial v_{i}}(D_{i}f_{\alpha}) + \frac{1}{2}\frac{\partial^{2}}{\partial v_{i}\partial v_{j}}(D_{ij}f_{\alpha}), \tag{35}$$

where D_i is the coefficient of dynamical friction and D_{ij} is the coefficient of diffusion in velocity space. They are defined as

$$D_{i} = \frac{\langle\triangle v_{i}\rangle}{\tau}, \tag{36}$$

$$D_{ij} = \frac{\langle\triangle v_{i}\triangle v_{j}\rangle}{\tau}, \tag{37}$$

where $\tau_1 \ll \tau \ll \tau_2$ is the time during which the coordinates of test particle do not change much but many fluctuations happen (τ_1 is the characteristic time of change of the velocity, τ_2 is the characteristic time of fluctuations).

When we evaluate change in velocity of a test particle we should take into account two forces acting on it. Besides force due to fluctuating electric field generated by all particles there is also force due to the polarization self-field induced by the test particle itself. The origin of the latter is the following. As test particle moves in plasma it polarizes plasma. The polarization cloud is highly correlated with the position of the test particle, which gives rise to the electric field acting on the test particle.

This polarization electric field can be found by subtracting the Coulomb electric field of the test particle itself from the total field:

$$\mathbf{E}_{pol}\left(\mathbf{r}=\mathbf{r}_T,t\right)=\mathbf{E}_{pol}\left(\mathbf{v}\right)=\mathrm{Re}\left[-4\pi q_T\int\frac{-i\mathbf{k}}{k^2}\left(\frac{1}{\varepsilon\left(\mathbf{k},\mathbf{kv}_T\right)}-1\right)\frac{d\mathbf{k}}{\left(2\pi\right)^3}\right]=$$
$$=4\pi q_T\int\frac{\mathbf{k}}{k^2}\mathrm{Im}\,\frac{1}{\varepsilon\left(\mathbf{k},\mathbf{kv}_T\right)}\frac{d\mathbf{k}}{\left(2\pi\right)^3}. \quad (38)$$

Since we are interested in the influence of this field on the test particle we have evaluated the polarization electric field at the position of test particle $\mathbf{r}=\mathbf{r}_T$.

Therefore the equation of motion of some test particle of type α is

$$\dot{\mathbf{v}}=\frac{q_\alpha}{m_\alpha}\mathbf{E}_{pol}\left(\mathbf{v}\right)+\frac{q_\alpha}{m_\alpha}\mathbf{E}\left[\mathbf{r}\left(t\right),t\right]. \quad (39)$$

Change in velocity is then (we choose current time to be zero, it obviously does not affect the expressions for the coefficients)

$$\triangle\mathbf{v}=\mathbf{v}\left(\tau\right)-\mathbf{v}\left(0\right)=\tau\frac{q}{m}\mathbf{E}_{pol}\left(\mathbf{v}\right)+\frac{q}{m}\int\limits_0^\tau\mathbf{E}\left[\mathbf{r}\left(t\right),t\right]dt, \quad (40)$$

and the position of the test particle is

$$\mathbf{r}\left(t\right)=\mathbf{r}_0\left(t\right)+\frac{q}{m}\int\limits_0^t dt_1\int\limits_0^{t_1}dt_2\mathbf{E}\left(\mathbf{r}\left(t_2\right),t_2\right), \quad (41)$$

where we have denoted the unperturbed orbit as

$$\mathbf{r}_0\left(t\right)\equiv\mathbf{r}\left(0\right)+\mathbf{v}\left(0\right)\cdot t. \quad (42)$$

Now we have everything to calculate the coefficients of the Fokker-Plank equation. Let us start with the calculation of the diffusion coefficient. Using (39) we can write the diffusion coefficient as

$$D_{ij}=\tau\frac{q_\alpha^2}{m_\alpha^2}E_{pol,i}E_{pol,j}+\frac{q_\alpha^2}{m_\alpha^2}\frac{1}{\tau}\int\limits_0^\tau dt\int\limits_0^\tau dt'\left\langle E_i\left[\mathbf{r}\left(t\right),t\right]E_j\left[\mathbf{r}\left(t'\right),t'\right]\right\rangle. \quad (43)$$

One can prove, however, that the contribution from polarization field is negligible. For example, according to Hubbard [1], since the velocity undergoes change within the time of order $\tau_1\ll\tau$ we have

$$\frac{q}{m}\left|\mathbf{E}_{total}\right| \sim \frac{v}{\tau_1}, \tag{44}$$

and, consequently,

$$\frac{q^2}{m^2}\left|E_{pol,i}E_{pol,j}\right|\tau \leq \frac{q^2}{m^2}\left|E_{tot}\right|^2\tau \sim v^2\frac{\tau}{\tau_1^2}. \tag{45}$$

On the other hand, $D_{ij} \sim \frac{v^2}{\tau_1} \gg v^2\frac{\tau}{\tau_1^2} \gtrsim \frac{q^2}{m^2}\left|E_{pol,i}E_{pol,j}\right|\tau$.

Moreover, because for the time τ the trajectories of the particles do not change dramatically we can approximate them with straight lines: $\mathbf{r}(t) \approx \mathbf{r}_0(t)$ so that the diffusion coefficient is simplified to

$$D_{ij} = \frac{q_\alpha^2}{m_\alpha^2}\frac{1}{\tau}\int\limits_0^\tau dt\int\limits_0^\tau dt'\left\langle E_i\left[\mathbf{r}_0(t),t\right]E_j\left[\mathbf{r}_0(t'),t'\right]\right\rangle, \tag{46}$$

or introducing the new variable $\xi = t - t'$

$$D_{ij} = \frac{q_\alpha^2}{m_\alpha^2}\frac{1}{\tau}\int\limits_0^\tau dt\int\limits_{t-\tau}^{t} d\xi\left\langle E_i\left[\mathbf{r}_0(t),t\right]E_j\left[\mathbf{r}_0(t-\xi),t-\xi\right]\right\rangle. \tag{47}$$

Because τ is much bigger than the time of fluctuations we can extend integration over $d\xi$ to infinity. Then integral over dt becomes trivial and we get

$$D_{ij} = \frac{q_\alpha^2}{m_\alpha^2}\int\limits_{-\infty}^{\infty} d\xi\left\langle E_i\left[\mathbf{r}_0(t),t\right]E_j\left[\mathbf{r}_0(t-\xi),t-\xi\right]\right\rangle. \tag{48}$$

Going to the Fourier space by means of (6) we get

$$D_{ij} = \frac{q_\alpha^2}{m_\alpha^2}\int\limits_{-\infty}^{\infty} d\xi d\xi e^{i(\omega_2-\mathbf{k}_2\mathbf{v})\xi}\int d\mathbf{r}\int dt\iint\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}\times$$
$$\iint\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}e^{-i(\omega_1+\omega_2)t}\left\langle E_i(\mathbf{k}_1,\omega_1)E_j(\mathbf{k}_2,\omega_2)\right\rangle. \tag{49}$$

After integration over dt, $d\mathbf{r}$ we get rid of one $\mathbf{k}$ and one ω, then integrating over $d\xi$ and using

$$\left\langle E_i(\mathbf{k},\omega)E_j(-\mathbf{k},-\omega)\right\rangle = \left\langle E_i(\mathbf{k},\omega)E_j^*(\mathbf{k},\omega)\right\rangle = \frac{k_ik_j}{k^2}\left\langle\mathbf{E}^2\right\rangle_{\mathbf{k},\omega} \tag{50}$$

we finally express the diffusion coefficient via the electric field fluctuation spectrum:

$$D_{ij} = \frac{2q_\alpha^2}{m_\alpha^2}\iint 2\pi\delta(\omega-\mathbf{kv})\frac{k_ik_j}{k^2}\left\langle\mathbf{E}^2\right\rangle_{\mathbf{k},\omega}\frac{d\mathbf{k}}{(2\pi)^3}\frac{d\omega}{2\pi}. \tag{51}$$

Now, let us calculate the coefficient of dynamical friction. To do this we again start with equation (39):

$$D_i(\mathbf{v}) = \frac{\langle\triangle\mathbf{v}\rangle}{\tau} = \frac{q_\alpha}{m_\alpha}\mathbf{E}_{pol,i}(\mathbf{v}) + \frac{q_\alpha}{m_\alpha}\frac{1}{\tau}\int\limits_0^\tau \langle\mathbf{E}_i[\mathbf{r}(t), t]\rangle\, dt. \tag{52}$$

Before we have evaluated such integral as the one in the equation above by ignoring effects of the fluctuating microfield on trajectories: $\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}\cdot t$. If we do this again this integral will give zero. However, since as it has been already demonstrated $\left\langle\mathbf{E}_{pol}^2\right\rangle \ll \left\langle\mathbf{E}^2\right\rangle$ we should keep the second order term in the integral, which arises from influence of the microfiled on trajectories:

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \frac{q}{m}\int\limits_0^t dt_1 \int\limits_0^{t_1} dt_2 \mathbf{E}(\mathbf{r}(t_2), t_2). \tag{53}$$

To simplify things we can Taylor expand the electric field:

$$\begin{aligned}\mathbf{E}[\mathbf{r}(t), t] = \mathbf{E}[\mathbf{r}_0(t), t] + \frac{\partial\mathbf{E}[\mathbf{r}_0(t), t]}{\partial\mathbf{r}_0}[\mathbf{r}(t) - \mathbf{r}_0] = \\ = \mathbf{E}[\mathbf{r}_0(t), t] + \frac{\partial\mathbf{E}[\mathbf{r}_0(t), t]}{\partial\mathbf{r}_0}\frac{q}{m}\int\limits_0^t dt_1\int\limits_0^{t_1} dt_2\mathbf{E}[\mathbf{r}_0(t_2), t_2].\end{aligned} \tag{54}$$

Hence the coefficient of dynamical friction becomes

$$\begin{aligned}D_i(\mathbf{v}) = \frac{\langle\triangle\mathbf{v}\rangle}{\tau} = \frac{q_\alpha}{m_\alpha}\mathbf{E}_{pol}(\mathbf{v}) + \frac{q_\alpha}{m_\alpha}\frac{1}{\tau}\int\limits_0^\tau \langle\mathbf{E}_i[\mathbf{r}_0(t), t]\rangle\, dt + \\ + \frac{q_\alpha^2}{m_\alpha^2}\frac{1}{\tau}\int\limits_0^\tau dt\int\limits_0^t dt_1\int\limits_0^{t_1} dt_2\left\langle\mathbf{E}_j[\mathbf{r}_0(t_2), t_2]\frac{\partial\mathbf{E}_i[\mathbf{r}_0(t), t]}{\partial\mathbf{r}_{j,0}}\right\rangle.\end{aligned} \tag{55}$$

The first integral as it has been already mentioned above is zero. To evaluate the second integral we start with integration by parts over t_1 and use of the fundamental theorem of calculus:

$$\frac{1}{\tau}\int_0^\tau dt\int_0^t dt_1\int_0^{t_1} dt_2\left\langle \mathbf{E}_j\left[\mathbf{r}_0\left(t_2\right),t_2\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle =$$

$$= \frac{1}{\tau}\int_0^\tau dt\left(t_1\int_0^{t_1} dt_2\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t_2\right),t_2\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle\right)\Bigg|_0^t -$$

$$-\frac{1}{\tau}\int_0^\tau dt\int_0^t dt_1\, t_1\frac{\partial}{\partial t_1}\int_0^{t_1} dt_2\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t_2\right),t_2\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle =$$

$$= \frac{1}{\tau}\int_0^\tau dt\int_0^t dt_2\, t\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t_2\right),t_2\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle -$$

$$-\frac{1}{\tau}\int_0^\tau dt\int_0^t dt_1\, t_1\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t_1\right),t_1\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle =$$

$$= \frac{1}{\tau}\int_0^\tau dt\int_0^t dt_1\ \left(t-t_1\right)\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t_1\right),t_1\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle \quad (56)$$

Now we again introduce the new variable $\xi = t - t_1$, and make upper limit in the integral over $d\xi$ go to infinity to get

$$\int_0^\infty d\xi\,\xi\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t-\xi\right),t-\xi\right]\frac{\partial \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]}{\partial \mathbf{r}_{0,j}}\right\rangle . \quad (57)$$

The function inside the integral depends only on ξ and our integral is equivalent to

$$\int_0^\infty d\xi\,\xi\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]\frac{\partial \mathbf{E}_j\left[\mathbf{r}_0\left(t+\xi\right),t+\xi\right]}{\partial \mathbf{r}_{0,j}}\right\rangle . \quad (58)$$

Writing $\mathbf{r}_0\left(t+\xi\right) = \mathbf{r}\left(t\right) + \mathbf{v}\cdot\xi$ we can change integration over $d\xi$ to integration over velocity and after comparison with (48) obtain

$$\frac{q_\alpha^2}{m_\alpha^2}\frac{\partial}{\partial \mathbf{v}_j}\int_0^\infty d\xi\left\langle \mathbf{E}_i\left[\mathbf{r}_0\left(t\right),t\right]\mathbf{E}_j\left[\mathbf{r}_0\left(t+\xi\right),t+\xi\right]\right\rangle = \frac{1}{2}\frac{\partial D_{ij}\left(\mathbf{v}\right)}{\partial v_j}. \quad (59)$$

Thus the coefficient of dynamical friction is

$$D_i\left(\mathbf{v}\right) = F_{pol,i}\left(\mathbf{v}\right) + \frac{1}{2}\frac{\partial D_{ij}\left(\mathbf{v}\right)}{\partial v_j}, \quad (60)$$

where we have denoted as $F_{pol,i}$ the contribution from polarization field to the dynamical friction coefficient. This contribution is easilty evaluated from (38)

$$\mathbf{F}_{pol}(\mathbf{v}) = \frac{q_\alpha}{m_\alpha}\mathbf{E}_{pol} = \frac{4\pi q_\alpha^2}{m_\alpha}\int \frac{\mathbf{k}}{k^2}\mathrm{Im}\,\frac{1}{\varepsilon(\mathbf{k},\mathbf{kv})}\frac{d\mathbf{k}}{(2\pi)^3} = \\ = \frac{4\pi q_\alpha^2}{m_\alpha}\int \frac{\mathbf{k}}{k^2}\frac{-\mathrm{Im}\,\varepsilon(\mathbf{k},\mathbf{kv})}{|\varepsilon(\mathbf{k},\mathbf{kv})|^2}\frac{d\mathbf{k}}{(2\pi)^3}. \quad (61)$$

Therefore we have expressed the coefficients of dynamical friction and the diffusion coefficient through the electric field fluctuation spectrum and through the dielectric function, both of which we already have calculated (formulas (22), (26)). It means that our goal of finding the relaxation equation for the average distribution function in plasma is achieved.

It is useful to rewrite the Fokker-Planck equation in the form where physical nature of the coefficients is separated:

$$\frac{\partial f_\alpha}{\partial t} = -\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{F}_{pol}(\mathbf{v}) f_\alpha(\mathbf{v}) - D_{fluct}(\mathbf{v})\frac{\partial f_\alpha(\mathbf{v})}{\partial \mathbf{v}}\right]. \quad (62)$$

Here we have introduced $D_{fluct,i}(\mathbf{v}) = \frac{1}{2}D_{ij}$. The physical origin of these coefficients is the following. Friction coefficient F_{pol} is an effective drag due to polarization, which decreases the energy of particles; while the diffusion coefficient D_{fluct} is due to correlation of the velocities of test particles and the fluctuating electric field, which, on average, results in increase of the energy of particles.

If we put the expression for dielectric function (26) in polarization term, and the expression for the fluctuation spectrum of electric field (22) into diffusion term we obtain

$$F_{pol}(\mathbf{v}) = \sum_\beta \frac{2n_\beta q_\alpha^2 q_\beta^2}{m_\alpha m_\beta}\iint \frac{\mathbf{kk}}{k^4}\frac{\delta(\mathbf{kv}-\mathbf{kv}')}{|\varepsilon(\mathbf{k},\mathbf{kv})|^2}\frac{\partial f_\beta(\mathbf{v}')}{\partial \mathbf{v}'}d\mathbf{k}d\mathbf{v}', \quad (63)$$

$$D_{ij}(\mathbf{v}) = \sum_\beta \frac{4n_\beta q_\alpha^2 q_\beta^2}{m_\alpha^2}\iint \frac{k_i k_j}{k^4}\frac{\delta(\mathbf{kv}-\mathbf{kv}')}{|\varepsilon(\mathbf{k},\mathbf{kv})|^2}f_\beta(\mathbf{v}')\,d\mathbf{k}d\mathbf{v}'. \quad (64)$$

Inserting these expressions into the relaxation equation we get

$$\frac{\partial f_\alpha}{\partial t} = \sum_\beta \frac{2n_\beta q_\alpha^2 q_\beta^2}{m_\alpha}\frac{\partial}{\partial \mathbf{v}}\iint \frac{\mathbf{kk}}{k^4}\frac{\delta(\mathbf{kv}-\mathbf{kv}')}{|\varepsilon(\mathbf{k},\mathbf{kv})|^2}\left[\frac{f_\beta(\mathbf{v}')}{m_\alpha}\frac{\partial f_\alpha(\mathbf{v})}{\partial \mathbf{v}} - \frac{f_\alpha(\mathbf{v})}{m_\beta}\frac{\partial f_\beta(\mathbf{v}')}{\partial \mathbf{v}'}\right]d\mathbf{k}d\mathbf{v}'. \quad (65)$$

This equation is called the Lenard-Balescu equatoin and was first derived by them independently in the early 1960s using different from the test particle model approach. We see that the collision integral on the right hand side of the Lenard-Balescu equation is very similar to the Landau collision integral with the exception that our integral is screened by the factor of $\left|\varepsilon\left(\mathbf{k},\mathbf{kv}\right)\right|^2$. It means that in plasma instead of collisions of bare particles with the Coulomb potential we have collisions of particles dressed in the polarization cloud. The Landau integral is divergent for both small and large $\mathbf{k}$, to eliminate divergence for large wavelengths the ad hoc cut off is usually introduced at $k_{min} = \frac{1}{\lambda_D}$. However, the Lenard-Balescu integral has divergence only for large k, while for small k divergence is absent due to screening and no cut off procedure is required.

Alternative derivation

It is useful to obtain the electric field fluctuation spectrum and the relaxation equation using different approach.

Let us start with microscopic distribution, which contains information about positions and velocities of all particles at any given time.

$$\mathcal{F}_\alpha\left(\mathbf{r},\mathbf{v},t\right) = \frac{1}{n_\alpha}\sum_{\alpha=1}^{N_\alpha}\delta\left(\mathbf{r}-\mathbf{r}_\alpha\left(t\right)\right)\delta\left(\mathbf{v}-\mathbf{v}_\alpha\left(t\right)\right), \tag{66}$$

where $n_\alpha = \frac{N_\alpha}{V}$ is average concentration and the sum is over all particles of given type. One can by direct differentiation make yourself sure that this function satisfies the so-called Klimontovich equation:

$$\frac{\partial\mathcal{F}_\alpha}{\partial t} + \mathbf{v}\frac{\partial\mathcal{F}_\alpha}{\partial\mathbf{r}} + \frac{q_\alpha}{m_\alpha}\mathbf{E}\frac{\partial\mathcal{F}_\alpha}{\partial\mathbf{v}} = 0, \tag{67}$$

where $\mathbf{E}$ is the total self-consistent microscopic electric field due to all plasma particles. In addition we also have the Poisson's equation for electric field:

$$\nabla\mathbf{E} = 4\pi\sum_\alpha q_\alpha n_\alpha\int\mathcal{F}_\alpha\left(\mathbf{r},\mathbf{v},t\right)d\mathbf{v}, \tag{68}$$

We define one-particle distribution function as average of (66) over Liouville distribution

$$f_\alpha\left(\mathbf{r},\mathbf{v},t\right) = \left\langle\mathcal{F}_\alpha\left(\mathbf{r},\mathbf{v},t\right)\right\rangle = \frac{N_\alpha}{n_\alpha}\int d\mathbf{r}_1\ldots d\mathbf{r}_{N_\alpha}d\mathbf{v}_1\ldots d\mathbf{v}_{N_\alpha}\mathcal{F}_\alpha\left(\mathbf{r},\mathbf{v},t\right)D_{N_\alpha}\left(\mathbf{r}_1,\ldots,\mathbf{r}_{N_\alpha},\mathbf{v}_1,\ldots,\mathbf{v}_{N_\alpha},t\right), \tag{69}$$

and we also define fluctuation of distribution function from average:

$$\delta f_\alpha = \mathcal{F}_\alpha\left(\mathbf{r},\mathbf{v},t\right) - f_\alpha\left(\mathbf{r},\mathbf{v},t\right). \tag{70}$$

Averaging equation (67) we get

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\langle \mathbf{E}\rangle \frac{\partial f_\alpha}{\partial \mathbf{v}} = -\frac{q_\alpha}{m_\alpha}\left\langle \delta\mathbf{E}\frac{\partial \delta f_\alpha}{\partial \mathbf{v}}\right\rangle, \tag{71}$$

$$\nabla\mathbf{E} = 4\pi\sum_\alpha n_\alpha q_\alpha \int f_\alpha(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}. \tag{72}$$

The total electric field is the sum of average collective filed and fluctuating field

$$\mathbf{E} = \langle \mathbf{E}\rangle + \delta\mathbf{E}.$$

We will consider the average collective field to be zero (taking into account average field will bring us to the equations of quasilinear theory). Thus we have kinetic equation for average distribution function

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}} = I_\alpha(\mathbf{v}), \tag{73}$$

with collision integral defined as

$$I_\alpha(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}}\left(\langle \mathbf{E}\delta f_\alpha\rangle\right). \tag{74}$$

We proceed by noticing that the fluctuation distribution function really consists of two distinctive parts. One is coherent response of plasma as a whole induced due to Coulomb interaction δf^c. Another part comes from discreetness of the particles and can be defined as follows

$$\delta\tilde{f_\alpha} = \mathcal{F}_\alpha(\mathbf{r},\mathbf{v},t) - \mathcal{F}_{\alpha,0}(\mathbf{r},\mathbf{v},t). \tag{75}$$

Here we introduced the microscopic distribution function in the absence of Coulomb interaction

$$\mathcal{F}_{\alpha,0}(\mathbf{r},\mathbf{v},t) = \frac{1}{n_\alpha}\sum_{\alpha=1}^{N_\alpha}\delta\left(\mathbf{r}-\mathbf{r}_{\alpha,0}(t)\right)\delta\left(\mathbf{v}-\mathbf{v}_{\alpha,0}(t)\right), \tag{76}$$

where $\mathbf{r}_{\alpha,0}(t) = \mathbf{r}_\alpha(t_0) + \mathbf{v}_\alpha(t)\cdot(t-t_0)$, $\mathbf{v}_{\alpha,0}(t) = \mathbf{v}_{\alpha,0}(t_0)$ are, respectively, the position and the velocity in the absence of interaction – just uniform motion along straight line.
Thus

$$\mathcal{F}_\alpha(\mathbf{r},\mathbf{v},t) = f_\alpha(\mathbf{r},\mathbf{v},t) + \delta f_\alpha = f_\alpha(\mathbf{r},\mathbf{v},t) + \delta f^{\alpha,c} + \delta\tilde{f}_\alpha. \tag{77}$$

Using a standard approach of the Fourier transforms defined according to (6) we can express coherent part of the fluctuation of the distribution function through the electric field fluctuation:

$$\delta f^{\alpha,c}_{\mathbf{k},\omega} = -i\frac{q}{m}\frac{\mathbf{E}_{\mathbf{k},\omega}}{\omega-\mathbf{k}\mathbf{v}}\frac{\partial f_\alpha}{\partial \mathbf{v}}. \tag{78}$$

At the same time we can write the Fourier component of the electric field as

$$\mathbf{E}_{\mathbf{k},\omega} = -i\frac{\mathbf{k}}{k^2}\frac{4\pi\sum_\alpha n_\alpha q_\alpha \int \delta\tilde{f}^\alpha_{\mathbf{k},\omega} d\mathbf{v}}{\varepsilon(\mathbf{k},\omega)}. \tag{79}$$

Thus the collision integral can be written as

$$I_\alpha(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial\mathbf{v}}\left\langle \int d\mathbf{r}\int dt \iint \mathbf{E}_{\mathbf{k}_1,\omega_1} e^{i\mathbf{k}_1\mathbf{r}-i\omega_1 t}\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\times \right.$$
$$\left. \iint\left(-i\frac{q_\alpha}{m_\alpha}\frac{\mathbf{E}_{\mathbf{k}_2,\omega_2}}{\omega_2-\mathbf{k}_2\mathbf{v}}\frac{\partial f_\alpha}{\partial\mathbf{v}} + \delta\tilde{f}^\alpha_{\mathbf{k}_2,\omega_2}\right) e^{i\mathbf{k}_2\mathbf{r}-i\omega_2 t}\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\omega_2}{2\pi}\right\rangle,$$

which after integration over dt and $d\mathbf{r}$ becomes

$$I_\alpha(\mathbf{v}) = -\frac{\partial}{\partial\mathbf{v}}\iint\frac{d\mathbf{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\left[-\frac{2q_\alpha^2}{m_\alpha^2}\left\langle\mathbf{E}^2\right\rangle_{\mathbf{k},\omega}\pi\delta(\omega-\mathbf{k}\mathbf{v})\frac{\partial f_\alpha}{\partial\mathbf{v}} + \frac{q_\alpha}{m_\alpha}\left\langle\mathbf{E}^*\delta\tilde{f}^\alpha(\mathbf{v})\right\rangle_{\mathbf{k},\omega}\right].$$

Correlation of two fluctuations of the distribution function related to discreetness of particles can be calculated using representation (75):

$$\left\langle\delta\tilde{f}^\alpha(\mathbf{r},\mathbf{v},t)\,\delta\tilde{f}^\beta(\mathbf{r}',\mathbf{v}',t')\right\rangle = \frac{\delta_{\alpha\beta}}{n_\alpha}\delta\left(\mathbf{r}_0(t)-\mathbf{r}'_0(t)\right)\delta\left(\mathbf{v}(t)-\mathbf{v}'(t)\right)f_\alpha(\mathbf{v}), \tag{80}$$

or in the Fourier space:

$$\left\langle\delta\tilde{f}^\alpha(\mathbf{v})\,\delta\tilde{f}^\beta(\mathbf{v}')\right\rangle_{\mathbf{k},\omega} = 2\pi\frac{\delta_{\alpha\beta}}{n_\alpha}\delta(\mathbf{v}-\mathbf{v}')\,\delta(\omega-\mathbf{k}\mathbf{v})\,f_\alpha(\mathbf{v}). \tag{81}$$

The correlation of the electric field and the fluctuating part of the distribution function is expressed through (81):

$$\left\langle\mathbf{E}^*\delta\tilde{f}^\alpha(\mathbf{v})\right\rangle_{\mathbf{k},\omega} = \sum_\beta i\frac{\mathbf{k}}{k^2}4\pi n_\beta q_\beta\int\frac{\left\langle\delta\tilde{f}^\alpha(\mathbf{v})\,\delta\tilde{f}^\beta(\mathbf{v}')\right\rangle_{\mathbf{k},\omega}}{\varepsilon(\mathbf{k},\omega)}d\mathbf{v}', \tag{82}$$

and hence

$$\mathrm{Re}\left\langle\mathbf{E}^*\delta\tilde{f}^\alpha(\mathbf{v})\right\rangle_{\mathbf{k},\omega} = -\frac{\mathbf{k}}{k^2}4\pi q_\alpha\frac{2\pi\delta(\omega-\mathbf{k}\mathbf{v})\,\mathrm{Im}\,\varepsilon(\mathbf{k},\omega)f_\alpha(\mathbf{v})}{|\varepsilon(\mathbf{k},\omega)|^2}. \tag{83}$$

From (79) the electric field fluctuation spectrum is

$$\left\langle\mathbf{E}^2\right\rangle_{\mathbf{k},\omega} = \frac{\mathbf{k}\mathbf{k}}{k^4}\frac{(4\pi)^2}{|\varepsilon(\mathbf{k},\omega)|^2}\sum_\alpha\sum_\beta n_\alpha q_\alpha n_\beta q_\beta\iint\left\langle\delta\tilde{f}^\alpha(\mathbf{v})\,\delta\tilde{f}^\beta(\mathbf{v}')\right\rangle_{\mathbf{k},\omega}d\mathbf{v}d\mathbf{v}', \tag{84}$$

which in the light of (81) reduces to

$$\left\langle \mathbf{E}^2 \right\rangle_{\mathbf{k},\omega} = \sum_\alpha \left(4\pi n_\alpha q_\alpha\right)^2 \frac{\mathbf{kk}}{k^4 \left|\varepsilon\left(\mathbf{k},\omega\right)\right|^2} \int \frac{2\pi}{n_\alpha} f_{\alpha 0}\left(\mathbf{v}\right) \delta\left(\omega - \mathbf{kv}\right) d\mathbf{v}. \tag{85}$$

The same expression for the electric field fluctuation spectrum as the expression (22) obtained previously from the test particle model.

Finally, inserting (83) and (85) into (73) we again get the Lenard-Balescu eqution for relaxation of the distribution function.

In conclusion, we see that this more rigorous approach yields the same results as the test particle model.

4 Scattering of radiation from plasma

Let us qualitatively discuss the key features of scattering of electromagnetic radiation from near equilibrium plasma. The quantitative analysis of scattering can be found in [2, 3].

It is known that homogeneous medium cannot scatter radiation. Scattering of radiation from plasma happens on plasma density fluctuations. If we know spectrum $\left\langle n^2 \right\rangle_{\mathbf{k},\omega}$ of the density fluctuations, then we are able to calculate the scattering cross section. This spectrum can be calculated using both the test particle model or the alternative approach described above.

We also know that a free electron scatters radiation much effectively than a free ion. So one might be tempted to attack the problem of scattering in plasma by considering the Thomson formula for scattering from a free electron and then consider scattering from plasma as an ensemble average over such electrons. Even though this approach works for the short wavelength radiation it produces results that contradict experimental data for scattering of long wavelength radiation. According to the experimental data long wavelength radiation appears to be scattered by ions: the width of scattering line is determined by the ion thermal velocity, while the scattering cross section is only half (if ions are singly charged) of what can be expected for scattering from free electrons.

This, at first glance surprising result, can be understood from the point of view of dressed particles. When a wave with wavelength larger than the Debye radius incidents on plasma it sees electrons and ions not as being free but together with their dresses. For example, electrons are screened by the polarization cloud with excess of ions and deprivation of electrons. In the presence of incident wave both central electron and the electrons of the dress oscillate in anti-phase such that radiation from them cancel each other and the total effective radiation in dipole approximation is essentially zero. The situation is different for the ion and its dress. The ion due to its massiveness is relatively motionless in the field of incident wave and, consequently, does not radiate much, but the electrons of its dress still oscillate and thus (re)radiate incident wave. The total radiation from such dressed ion is nonzero. Notice, however, that the momentum and energy of the incident wave is transferred to the central ion (and to the scattered wave, of course) and not to the electrons of the dress, which are rather the intermediate step in scattering. Therefore we can say that in plasma due to collective effects radiation is scattered mainly by ions. Another important aspect of scattering from plasma is that the above-mentioned screening is dynamical, i.e. depends on the velocities of the particles. For example, electrons with velocities much larger than thermal velocity are not screened (see section #1) and thus scatter as effectively as free electrons. However, since only small fraction of electrons from the tail of Maxwell distribution have such big velocities their effect on the total scattering cross section of radiation from plasma is minor.

In the light of the alternative derivation from the previous section the concept of the test particle can be understood as the averaging of motion of particles over fluctuations. The motion of every particle in plasma can be separated into two components: one is average motion and the other is fluctuating motion. When plasma particle performs the role of scattering center it performs average motion, when particle is part of the dress of some other particle it performs fluctuating motion.

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